Table 1 Optimum area ($\bar{A}$) and temperature ($\bar{T}$) distribution along length of the cooling fin – five-element solution ($\bar{T}_c = 0.3, N_G = 1.0, B_i = 0.0$)

	$\bar{A}$			$\bar{T}$		
Node	Present method	Exact solution[6]	Ref. 4	Present method	Exact solution[6]	Ref. 4
1[a]	2.2263[c]	2.2222	1.9464[d]	0.0	0.0	0.0
2	1.9915	1.9876	1.7846	0.0853	0.0853	0.0930
3	1.7281	1.7213	1.5938	0.1604	0.1650	0.1715
4	1.4092	1.4054	1.3377	0.2239	0.2241	0.2343
5	1.0271	0.9938	0.9213	0.2727	0.2732	0.2790
6[b]	0.3055	0.0		0.3000	0.3000	0.3000
$\bar{V}_{opt}$	1.4845	1.4815	1.5168			

[a]Node at $\bar{T} = 0$. [b]Node at insulated end. [c]Nodal areas of linearly tapered elements (solution at 25 interations). [d]Constant area elements (solution at 20 iterations).

Table 2 Effect of convection on minimum volume for cooling fin – five-element solution ($\bar{T}_c = 0.3, \bar{T}_a = 0.05, N_G = 1.0$)

	$\bar{V}$	
B_i	Present method	Constant area elements (Ref. 4)[c]
0.0[a]	1.4845 (25)[b]	1.5168 (1.5023)[d]
0.05	1.4564 (25)	1.4878 (1.4736)
0.2	1.3712 (30)	1.4005 (1.3870)
0.5	1.1985 (34)	1.2232 (1.2115)

[a]For $B_i = 0.0$, $\bar{V}_{exact} = 1.4815$ (Ref. 6). [b]Number of iterations to obtain convergence. [c]Results at 20 iterations. [d]Numbers in the brackets represent 20-element solution.

iterative process is terminated when the constraint temperature converges up to four significant figures.

Table 1 gives the nondimensional area distribution $\bar{A}$ ($\bar{A} = A/A_o$, where A_o is the reference area) and temperature distribution $\bar{T}$ at the six nodes of the fin for the heat generation number, $N_G = 1.0 (N_G = qL^2/KA_oT_o$, where K is the thermal conductivity of the material), without convective heat loss, i.e., the Biot number, $B_i = 0 (B_i = hL^2/KA_o$, where h is the convective heat-transfer coefficient). The exact solution for this case[6] and the solution obtained with constant area elements with element areas as design variables[4] are also presented in this table. Nondimensional optimum fin volume $\bar{V}_{opt}$ ($\bar{V}_{opt} = V/A_oL$, where V is the volume of the fin) is given in this table. The present solution converged in 25 iterations and the solution of Ref. 4 is given at 20 interations. It can be seen from Table 1 that the results obtained with nodal areas as design variables agree well with the exact solution, whereas those obtained with element areas as design variables are inferior and give higher fin volume compared to the present solution.

In Table 2, the effect of convective heat loss on the minimum mass design is shown. In this table, the nondimensional volume $\bar{V}$ is presented for various values of B_i (the ambient temprature $\bar{T}_a$ is taken as 0.05). $\bar{V}$ obtained from the present five-element solution is less compared to the 20-element solution of Ref. 4. Also it can be noted here that as B_i increases, the number of iterations required for convergence increases. This is because the convection matrix is independent of design variables. From this table it can be concluded that the effect of convective heat loss is to decrease the optimum volume.

Conclusion

The concept of nodal variable as a design variable in the optimality criterion approach for obtaining minimum mass design of structures is proposed. The effectiveness of this proposal is shown through an example of obtaining the minimum mass (volume if density is constant) design of a cooling fin subjected to a temperature constraint.

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Amplification of Finite-Amplitude Waves in a Radiating Gas

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Introduction

USING the theory of singular surfaces, a number of problems relating to wave propagation in diverse branches of continuum mechanics have been studied previously.[1-7] The present paper uses the singular surface theory of Thomas[6] to study the propagation of arbitrarily shaped finite-amplitude waves in a radiating gas near the optically thin limit. The gas is assumed to be perfect, optically gray, and in thermodynamic equilibrium. It is found that a

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compressive wave decays or takes a stable wave form, or terminates into a shock after a finite time, depending on whether the initial wave amplitude is less than, equal to, or exceeds a critical value. It is shown that the geometry of the wave affects the growth properties only indirectly in that the critical amplitude depends on the initial curvature of the wave front. Application is made to waves of plane, cylindrical, and spherical geometry. It is found that both radiative flux and wave front curvature have a delaying effect on the formation of a shock wave.

The basic equations appropriate to the problem under consideration are

$$(\partial\rho/\partial t)+u_i\rho,_i+\rho u_{i,i}=0 \tag{1}$$

$$\rho(\partial u_i/\partial t)+\rho u_j u_{i,j}+p,_i=0 \tag{2}$$

$$(\partial p/\partial t)+u_i p,_i+\gamma p u_{i,i}+Q(\gamma-1)=0 \tag{3}$$

where p is the gas pressure, ρ the density, γ the ratio of specific heats, and u_i the gas velocity components. The quantity $Q=4k_p\sigma T^4$ is the rate of energy loss by the gas per unit volume through radiation, where k_p, σ, and T are the Planck mean absorption coefficient, Stefan-Boltzmann constant, and the gas temperature, respectively. Here the summation convention on repeated indices is employed and a comma followed by an index (i) denotes a partial derivative with respect to space variable x_i. Throughout this paper we shall take the spectrum of the suffixes i,j,k to be 1,2,3.

Velocity of Wave Propagation

Let us consider a propagating singular surface $S(t)$ across which the velocity vector u_i is continuous but $\partial u_i/\partial t$ suffers a finite-jump discontinuity; such a singularity surface is called a weak wave, the vector amplitude of which is defined by

$$a_i=[\partial u_i/\partial t] \tag{4}$$

where the brackets denote jump across S in the quantity enclosed. Let us denote the unit normal vector by n_i and the normal speed of advance of S by $G>0$. The integral forms of Eqs. (1) and (2) imply that across S

$$[\rho]=0,\quad [p]=0 \tag{5}$$

Here, we have assumed that the relative speed of advance of S in the fluid does not vanish; this means that S is not a material surface. In view of the equation of state $p=\rho RT$ and Eq. (5), we infer that T is also continuous across S. The first-order geometric and kinematic compatibility conditions[8] for the wave surface S reduce to

$$[Z,_i]=[Z,_j]n_i n_j;\quad [\partial Z/\partial t]=-G[Z,_j]n_j \tag{6}$$

where Z may represent any of the flow quantities p,ρ,T, and u_i. Equation (4) in view of Eq. (6) yields

$$a_i=-G\lambda_i \tag{7}$$

where $\lambda_i=[u_{i,j}]n_j$ are the quantities defined on S.

Forming jumps across S in Eqs. (1-3) and using Eqs. (4-7), we get

$$G(G-u_n)\zeta+a_i n_i\rho=0 \tag{8}$$

$$\rho(G-u_n)a_i+G\xi n_i=0 \tag{9}$$

$$G(G-u_n)+a_i n_i\rho c^2=0 \tag{10}$$

where $\zeta=[\rho,_j]n_j$, $\xi=[p,_j]n_j$, $u_n=u_i n_i$, and $c=(\gamma p/\rho)^{1/2}$ are the quantities defined on S. Equation (9) shows that a_i is parallel to n_i, which leads to the conclusion that for the type of material we are considering only longitudinal waves are possible. Keeping in view that S is not a material surface, Eqs. (9) and (10) yield $G-u_n=\pm c$. For an advancing wave S, the relative speed of propagation has to be positive, and so we take

$$G-u_n=c \tag{11}$$

We shall be concerned in the rest of the paper with the situation in which the medium ahead of the wave is uniform and at rest, so that the successive positions of S at different instants form a family of parallel surfaces with straight lines as their orthogonal trajectories. Equations (8, 9, and 11) then yield

$$a_i=an_i \tag{12}$$

$$a=-\zeta c^2/\rho=-\xi/\rho \tag{13}$$

Derivation of Wave Amplitude

When Eqs. (2) and (3) are differentiated with respect to x_k and jumps are taken across S, we find on using the second-order compatibility conditions[8] and the foregoing results that

$$(\delta a/\delta t)=(c/\rho)(\tilde{\xi}-\rho c\tilde{\lambda})$$

$$(\delta\xi/\delta t)=c(\tilde{\xi}-\rho c\tilde{\lambda})$$
$$+(2c\Omega-4\rho^{-1}c^{-2}(\gamma-1)^2Q)\xi-\xi^2(\gamma+1)(\rho c)^{-1}$$

where $\delta/\delta t$ is the time derivative along an orthogonal trajectory of S, $\tilde{\lambda}=[u_{i,jk}]n_i n_j n_k$, and $\tilde{\xi}=[p,_{jk}]n_j n_k$ are the quantities defined on S, and $\Omega=\frac{1}{2}g^{\alpha\beta}b_{\alpha\beta}$ is the mean curvature of S with $g^{\alpha\beta}$ and $b_{\alpha\beta}$ being, respectively, the first and second fundamental forms of S (the indices α and β, which refer to surface coordinates, take the values 1 and 2). For S, propagating with constant velocity c, the mean curvature has the representation[9] $\Omega=(\bar{\Omega}-ct\bar{K})/(1-2\bar{\Omega}ct+c^2t^2\bar{K})$, where $\bar{\Omega}$ and $\bar{K}$ are the mean and Gaussian curvatures of S at $t=0$, respectively. The above two equations, on eliminating the factor $(\tilde{\xi}-\rho c\tilde{\lambda})$ and using Eq. (13), yield

$$(\delta a/\delta t)-(c\Omega-\mu)-(2c)^{-1}(\gamma+1)a^2=0 \tag{14}$$

where $\mu=2(\gamma-1)^2Q/(\rho c^2)$ is a positive quantity. Equation (14) is the governing equation for the growth or decay of the wave amplitude which we have been seeking. Equation (14) can be integrated to yield

$$a=\frac{\bar{a}(1-2\bar{\Omega}ct+\bar{K}c^2t^2)^{-1/2}\exp(-\mu t)}{1-(2c)^{-1}(\gamma+1)\bar{a}\int_0^t(1-2\bar{\Omega}c\tau+\bar{K}c^2\tau^2)^{-1/2}e^{-\mu\tau}d\tau} \tag{15}$$

where $\bar{a}$ is the value of a at $t=0$.

Discussion

Here we are interested in studying only expanding (i.e., outward traveling) waves; for such waves $\bar{\Omega}\leq 0$ and $\bar{K}\geq 0$. Further, since $\mu>0$, it is a simple matter to show that the integral in the denominator of Eq. (15) converges to a positive finite limit as $t\rightarrow\infty$. Thus, it follows from Eq. (15) that if $\bar{a}<0$ (i.e., an expansion wave front), then $a\rightarrow 0$ as $t\rightarrow\infty$; this means that the wave decays and ultimately damps out. But if $\bar{a}>0$ (i.e., a compression wave front), then there exists a finite positive critical initial wave amplitude given by

$$\bar{a}=\left\{(2c)^{-1}(\gamma+1)\int_0^\infty(1-2\bar{\Omega}ct+\bar{K}c^2t^2)^{-1/2}e^{-\mu t}dt\right\}^{-1} \tag{16}$$

such that when $\bar{a}<\hat{a}$, $a\rightarrow 0$ as $t\rightarrow\infty$; i.e., such a compression wave decays and ultimately damps out. When $\bar{a}=\hat{a}$, using L'Hospital's and Leibnitz's rules, it transpires that $a\rightarrow 2\mu c(\gamma+1)^{-1}$ as $t\rightarrow\infty$, i.e., such a compression wave can neither terminate in a shock wave nor can it ever damp out; in fact it ultimately takes a stable wave form. But when $\bar{a}>\hat{a}$, then $a\rightarrow\infty$ as $t\rightarrow\hat{t}$ (i.e., the wave terminates in a shock wave in a finite time $\hat{t}$), where $\hat{t}$ is given by the solution of

$$\int_0^{\hat{t}} (1-2\bar{\Omega}ct+\bar{K}c^2t^2)e^{-\mu t}\mathrm{d}t=2c((\gamma+1)\bar{a})^{-1} \tag{17}$$

Further, it can be seen from Eq. (17) that $\partial\hat{t}/\partial\mu>0$, which means that an increase in μ causes an increase in $\hat{t}$. Thus we conclude that the radiative transfer effects cause the shock formation time to increase.

One may verify from Eqs. (16) and (17) that for plane waves ($\bar{\Omega}=0=\bar{K}$), the initial critical amplitude and the shock formation time are given by

$$\hat{a}=2\mu c(\gamma+1)^{-1}, \quad \hat{t}=\mu^{-1}\ln(1-(\hat{a}/\bar{a}))^{-1}$$

Further, if the wave front S at $t=0$ is a cylinder (sphere) of radius $\bar{R}$, then at any time $t>0$, S is a cylinder (sphere) of radius $R=\bar{R}+ct$. Then it follows from Eq. (16) that for cylindrical ($\bar{\Omega}=-1/(2\bar{R})$, $\bar{K}=0$) and spherical ($\bar{\Omega}=-1/\bar{R}$, $\bar{K}=(\bar{R})^{-2}$) waves, the critical initial wave amplitudes are given respectively, by

$$\hat{a}=\frac{2c}{(\gamma+1)}\left(\frac{\mu c}{\pi\bar{R}}\right)^{1/2}\left[\frac{\exp(-\mu\bar{R}/c)}{\mathrm{erfc}(\mu\bar{R}/c)^{1/2}}\right] \tag{18}$$

and

$$\hat{a}=2c^2\{(\gamma+1)\bar{R}E_1(\mu\bar{R}/c)\}^{-1}\exp(-\mu\bar{R}/c) \tag{19}$$

where

$$\mathrm{erfc}(x)=(2/\sqrt{\pi})\int_x^\infty e^{-t^2}\mathrm{d}t \text{ and } E_1(x)=\int_x^\infty t^{-1}e^{-t}\mathrm{d}t$$

are the tabulated functions known as the complementary error function and the exponential integral respectively. Since $\mathrm{erfc}(x)<(x\sqrt{\pi})^{-1}\exp(-x^2)$, and $E_1(x)<x^{-1}e^{-x}$, it follows from Eqs. (18) and (19) that $\hat{a}\rightarrow 2\mu c(\gamma+1)^{-1}$, which means that the critical initial amplitude for a cylindrical or spherical wave is greater than that for a plane wave. Also, for cylindrical and spherical waves Eq. (17) shows that $\partial\hat{t}/\partial\bar{R}<0$, which means that an increase in the initial wave front curvature causes a delay in the formation of a shock wave.

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